

# A REINTERPRETATION OF EMERTON'S $p$ -ADIC BANACH SPACES

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ABSTRACT. It is shown that the  $p$ -adic Banach spaces introduced by Emerton are isomorphic to the cohomology groups of the sheaf of continuous  $\mathbb{Q}_p$ -valued functions on a certain space. Some applications of this result are discussed.

## 1. INTRODUCTION

Let  $\mathbb{G}$  be a reductive group over a number field  $k$ . We fix once and for all a maximal compact subgroup  $K_\infty \subset \mathbb{G}(k \otimes \mathbb{R})$ , and we consider the “Shimura manifolds”:

$$Y(K_f) = \mathbb{G}(k) \backslash \mathbb{G}(\mathbb{A}) / K_\infty^o K_f,$$

where  $K_f$  is a compact open subgroup of  $\mathbb{G}(\mathbb{A}_f)$ , and  $K_\infty^o$  is the identity component in  $K_\infty$ . Fix once and for all a finite prime  $\mathfrak{p}$  of  $k$ , and let  $p$  be the rational prime below  $\mathfrak{p}$ . In [2] Emerton introduced the following spaces:

$$\tilde{H}_*(K^\mathfrak{p}, \mathbb{Z}_p) = \lim_{\leftarrow s} \lim_{\rightarrow K_\mathfrak{p}} H_*(Y(K_\mathfrak{p} K^\mathfrak{p}), \mathbb{Z}/p^s),$$

$$\tilde{H}_*(K^\mathfrak{p}, \mathbb{Q}_p) = \tilde{H}_*(K^\mathfrak{p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

where  $*$  is either the empty symbol, denoting usual cohomology, or “c”, denoting compactly supported cohomology. Here the  $K_\mathfrak{p}$  ranges over the compact open subgroups of  $\mathbb{G}(k_\mathfrak{p})$ , and  $K^\mathfrak{p}$  is a fixed compact open subgroup of  $\mathbb{G}(\mathbb{A}_f^\mathfrak{p})$ . The spaces  $H_*(K^\mathfrak{p}, \mathbb{Q}_p)$  are  $p$ -adic Banach spaces, and are central to Emerton’s construction of eigenvarieties in [2].

The aim of this paper is to give a more convenient interpretation of these spaces. To explain this interpretation, consider the topological space:

$$Y(K^\mathfrak{p}) = \mathbb{G}(k) \backslash \mathbb{G}(\mathbb{A}) / K^\mathfrak{p} K_\infty^o = \lim_{\leftarrow K_\mathfrak{p}} Y(K^\mathfrak{p} K_\mathfrak{p}).$$

Let  $\mathcal{C}_{\mathbb{Z}_p}$  (resp.  $\mathcal{C}_{\mathbb{Q}_p}$ ) be the sheaf of continuous  $\mathbb{Z}_p$ -valued (resp.  $\mathbb{Q}_p$ -valued) functions on  $Y(K^\mathfrak{p})$ . The space  $Y(K^\mathfrak{p})$  need not be compact, but it is homotopic to a profinite simplicial complex, which we shall call  $Y(K^\mathfrak{p})^{\text{B.S.}}$ . We shall also use the notation  $\partial Y(K^\mathfrak{p})^{\text{B.S.}} = Y(K^\mathfrak{p})^{\text{B.S.}} \setminus Y(K^\mathfrak{p})$ .

Our main result is the following.

**Theorem 1.** *There are canonical isomorphisms*

$$\tilde{H}^\bullet(K^\mathfrak{p}, \mathbb{Z}_p) = \check{H}^\bullet(Y(K^\mathfrak{p}), \mathcal{C}_{\mathbb{Z}_p}), \quad \tilde{H}_c^\bullet(K^\mathfrak{p}, \mathbb{Z}_p) = \check{H}_c^\bullet(Y(K^\mathfrak{p})^{\text{B.S.}}, \partial Y(K^\mathfrak{p}), \mathcal{C}_{\mathbb{Z}_p}),$$

and similarly for  $\mathbb{Q}_p$ . The right hand side of these equations is Čech cohomology, which in this case is equal to sheaf cohomology.

One of our aims in proving this result is to compare the spaces  $\tilde{H}^\bullet$  and  $\tilde{H}_c^\bullet$ . An immediate consequence of our result is a long exact sequence involving these two spaces

$$\rightarrow \tilde{H}_c^n(K^{\mathfrak{p}}, \mathbb{Z}_p) \rightarrow \tilde{H}^n(K^{\mathfrak{p}}, \mathbb{Z}_p) \rightarrow \tilde{H}_\partial^n(K^{\mathfrak{p}}, \mathbb{Z}_p) \rightarrow \tilde{H}_c^{n+1}(K^{\mathfrak{p}}, \mathbb{Z}_p) \rightarrow,$$

where we are using the notation

$$\tilde{H}_\partial^n(K^{\mathfrak{p}}, \mathbb{Z}_p) = \check{H}^n(\partial Y(K^{\mathfrak{p}})^{\text{B.S.}}, \mathbb{C}_{\mathbb{Z}_p}).$$

This is significant, since one can show that  $\tilde{H}_\partial^n$  vanishes unless  $n$  is quite small. For example, if  $\mathbb{G}$  has real rank 1 and  $\mathfrak{p}$  is the only prime of  $k$  above  $p$ , then only  $\tilde{H}_\partial^0$  is non-zero. As a consequence, we know that for such groups,  $\tilde{H}^n$  and  $\tilde{H}_c^n$  are equal for  $n \geq 2$ . Generalizations of such results will be discussed in a forthcoming paper.

It is also envisioned that these results should give new insight into Eisenstein cohomology classes. To see why this might be the case, we recall that Emerton proved a spectral sequence

$$\text{Ext}_{\mathfrak{g}}^p(\check{W}, \tilde{H}_*^q(K^{\mathfrak{p}}, k_{\mathfrak{p}})_{\text{loc.an.}}) \implies H_*^{p+q}(Y(K^{\mathfrak{p}}), W),$$

where  $W$  is a local system on  $Y(K^{\mathfrak{p}})$  given by a finite dimensional representation of  $\mathbb{G}$  over  $k_{\mathfrak{p}}$ . We remark that if  $\mathbb{G}$  is semi-simple then  $\tilde{H}_c^n$  is zero for  $n < \text{rank}_k(\mathbb{G})$ ; this follows easily by Poincaré duality. On the other hand  $\tilde{H}_\partial^n$  vanishes for all but small values of  $n$ . Hence one might expect to be able to recover the boundary cohomology quite low down in the filtration given by the spectral sequence.

## 2. SOME FACTS ABOUT ČECH COHOMOLOGY

Let  $X$  be a topological space and  $\mathcal{F}$  a presheaf on  $X$ . For an open cover  $\mathfrak{U} = \{U_i : i \in I\}$  of  $X$ , we define the Čech complex  $\check{C}^\bullet(\mathfrak{U}, \mathcal{F})$  by

$$\check{C}^n(\mathfrak{U}, \mathcal{F}) = \{(f_{i_0, \dots, i_n})_{i_0, \dots, i_n \in I^{n+1}} : f_{i_0, \dots, i_n} \in \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_n})\}.$$

The cohomology groups of this complex are written  $\check{H}^\bullet(\mathfrak{U}, \mathcal{F})$ . The Čech cohomology groups are defined to be the direct limits of these cohomology groups:

$$\check{H}^n(X, \mathcal{F}) = \varinjlim_{\mathfrak{U}} \check{H}^n(\mathfrak{U}, \mathcal{F}).$$

In fact  $\check{H}^n(X, \mathcal{F})$  depends only on the sheafification of  $\mathcal{F}$ .

**Theorem 2** (Leray's Theorem). *Let  $\mathcal{F}$  be a sheaf on a topological space  $X$  and  $\mathfrak{U}$  a countable open cover of  $X$ . If  $\mathcal{F}$  is acyclic on every finite intersection of elements of  $\mathfrak{U}$ , then*

$$\check{H}^n(X, \mathcal{F}) = \check{H}^n(\mathfrak{U}, \mathcal{F}).$$

**Theorem 3** (Thm. III.4.12 of [1]). *If  $\mathcal{F}$  is a sheaf on  $X$  and  $X$  is paracompact, then the Čech cohomology groups of  $\mathcal{F}$  are equal to its sheaf cohomology groups, i.e. the derived functors of the global sections functor.*

Given a presheaf  $\mathcal{F}$  on a topological space  $Y$ , and a subspace  $Z \subset Y$ , we define presheafs  $\mathcal{F}_Z$  and  $\mathcal{F}^Z$  on  $X$  by

$$\mathcal{F}_Z(U) = \begin{cases} \mathcal{F}(U) & \text{if } U \cap Z \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{F}^Z(U) = \begin{cases} 0 & \text{if } U \cap Z \neq \emptyset \\ \mathcal{F}(U) & \text{otherwise.} \end{cases}$$

It turns out that  $\check{H}^\bullet(Z, \mathcal{F}) = \check{H}^\bullet(X, \mathcal{F}_Z)$ , and one defines

$$\check{H}^\bullet(Y, Z, \mathcal{F}) = \check{H}^\bullet(Y, \mathcal{F}^Z).$$

There is a short exact sequence of presheafs:

$$0 \rightarrow \mathcal{F}^A \rightarrow \mathcal{F} \rightarrow \mathcal{F}_A \rightarrow 0,$$

This gives a long exact sequence:

$$\check{H}^n(\mathfrak{U}, \mathcal{F}^A) \rightarrow \check{H}^n(\mathfrak{U}, \mathcal{F}) \rightarrow \check{H}^n(\mathfrak{U}, \mathcal{F}_A) \rightarrow \check{H}^{n+1}(\mathfrak{U}, \mathcal{F}^A).$$

Passing to the direct limit, we obtain the long exact sequence of Čech cohomology groups:

$$\check{H}^n(X, A, \mathcal{F}) \rightarrow \check{H}^n(X, \mathcal{F}) \rightarrow \check{H}^n(A, \mathcal{F}) \rightarrow \check{H}^{n+1}(X, A, \mathcal{F}).$$

If  $A$  is an abelian group, then we shall also write  $\underline{A}$  for the sheaf of locally constant  $A$ -valued functions. Using Leray's theorem, one easily proves the following:

**Theorem 4** (Comparison Theorem). *Let  $Y$  be a finite simplicial complex, and  $Z \subset Y$  a subcomplex. For any abelian group  $A$ , we have*

$$\check{H}^\bullet(Y, Z, \underline{A}) = H^\bullet(Y, Z, A),$$

where the right hand side is singular cohomology.

In fact the comparison theorem holds for much more general topological spaces (see for example [3]).

**Theorem 5** (Lem. 6.6.11, Cor 6.1.11 and Cor. 6.9.9 of [3]). *Let  $Y$  be a finite simplicial complex and  $Z$  a subcomplex. For any abelian group  $A$ , we have*

$$H_c^\bullet(Y \setminus Z, A) = H^\bullet(Y, Z, A).$$

### 3. PROOFS

Emerton used the following formalism to introduce the groups  $\tilde{H}_*^n$ . Let  $G$  be a compact,  $\mathbb{Q}_p$ -analytic group, and fix a basis of open, normal subgroups:

$$G = G_0 \supset G_1 \supset \dots$$

Suppose we have a sequence of simplicial maps between finite simplicial complexes

$$\dots \rightarrow Y_2 \rightarrow Y_1 \rightarrow Y_0,$$

and subcomplexes:

$$\dots \rightarrow Z_2 \rightarrow Z_1 \rightarrow Z_0,$$

each equipped with a right action of  $G$ , and satisfying the following conditions:

- (1) the maps in the sequence are  $G$ -equivariant;
- (2)  $G_r$  acts trivially on  $Y_r$ .
- (3) if  $0 \leq r' \leq r$  then the maps  $Y_r \rightarrow Y_{r'}$  and  $Z_r \rightarrow Z_{r'}$  are Galois covering maps with deck transformations provided by the natural action of  $G_{r'}/G_r$  on  $Y_r$ .

Given this data, we let  $Y$  be the projective limit of the spaces  $Y_r$ , and  $Z$  be the projective limit of the spaces  $Z_r$ . We shall use the notation  $Y^0 = Y \setminus Z$ ,  $Y_i^0 = Y_i \setminus Z_i$ . Emerton defined the following spaces:

$$\tilde{H}_c^n(Y^0, \mathbb{Z}_p) = \varprojlim_s \varinjlim_r H_c^n(Y_r^0, \mathbb{Z}/p^s), \quad \tilde{H}_c^n(Y^0, \mathbb{Q}_p) = \tilde{H}_c^n(Y^0, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

In applications,  $G$  will be a compact open subgroup of  $\mathbb{G}(k_p)$ , and the space  $Y$  will be either  $Y(K^p)^{\text{B.S.}}$  or  $\partial Y^{\text{B.S.}}$ . If  $Y = Y(K^p)^{\text{B.S.}}$  then we may use the subspace  $Z = \partial Y(K^p)^{\text{B.S.}}$ .

**Theorem 6.** *With the above notation,*

$$\tilde{H}_c^n(Y^0, \mathbb{Z}_p) = \check{H}^n(Y, Z, \mathcal{C}_{\mathbb{Z}_p}), \quad \tilde{H}_c^n(Y^0, \mathbb{Q}_p) = \check{H}^n(Y, Z, \mathcal{C}_{\mathbb{Q}_p}).$$

*Proof.* We shall prove the case with coefficients in  $\mathbb{Z}_p$ . The  $\mathbb{Q}_p$  case is a consequence. We shall write  $\mathcal{C}$  instead of  $\mathcal{C}_{\mathbb{Z}_p}$ . To prove the theorem, we construct an acyclic cover of  $Y$  and apply Leray's Theorem.

**3.1. A cover.** We first choose a finite open cover  $\mathfrak{U}$  of  $Y_0$  with the following properties:

- (1) If  $U$  is an intersection of finitely many sets in  $\mathfrak{U}$  then either  $U$  is empty or  $U$  is contractible.
- (2) If  $U$  is an intersection of finitely many sets in  $\mathfrak{U}$  and  $U \cap Z_0$  is non-empty, then  $U \cap Z_0$  is a deformation retract of  $U$ .

For each  $U \in \mathfrak{U}$ , we let  $U^{(r)}$  be the preimage of  $U$  in  $Y_r$ . The sets  $U^{(r)}$  form an open cover  $\mathfrak{U}^{(r)}$  of  $Y_r$ , and have the following properties:

- (1) For every  $U_1^{(r)}, \dots, U_s^{(r)} \in \mathfrak{U}^{(r)}$  with non-empty intersection, the intersection  $U_1^{(r)} \cap \dots \cap U_s^{(r)}$  is isomorphic as a topological  $G$ -set to  $(U_1 \cap \dots \cap U_r) \times (G/G_r)$ . In particular, the intersection is homotopic to a finite set.
- (2) If  $U_1^{(r)}, \dots, U_s^{(r)} \in \mathfrak{U}^{(r)}$  and  $U_1^{(r)} \cap \dots \cap U_s^{(r)} \cap Z_r$  is non-empty, then  $U_1^{(r)} \cap \dots \cap U_s^{(r)} \cap Z_r$  is a deformation retract of  $U_1^{(r)} \cap \dots \cap U_s^{(r)}$ .

Furthermore, for each set  $U \in \mathfrak{U}$ , we define  $\tilde{U}$  to be the preimage of  $U$  in  $Y$ . The sets  $\tilde{U}$  form an open cover  $\tilde{\mathfrak{U}}$  of  $Y$ . We immediately verify the following:

- (1) if  $\tilde{U}_1, \dots, \tilde{U}_s \in \tilde{\mathfrak{U}}$  have non-empty intersection, then their intersection is equivalent as a topological  $G$ -set to  $(U_1 \cap \dots \cap U_s) \times G$ .
- (2) if  $\tilde{U}_1, \dots, \tilde{U}_s \in \tilde{\mathfrak{U}}$  and  $\tilde{U}_1 \cap \dots \cap \tilde{U}_s \cap Z$  is non-empty, then  $\tilde{U}_1 \cap \dots \cap \tilde{U}_s \cap Z$  is a deformation retract of  $\tilde{U}_1 \cap \dots \cap \tilde{U}_s$ .

**3.2.  $\mathfrak{U}^{(r)}$  is  $(\mathbb{Z}/p^s)^{Z_r}$ -acyclic.** Let  $U$  be an intersection of finitely many sets in  $\mathfrak{U}$ , and let  $U^{(r)}$  be the preimage of  $U$  in  $Y_r$ . We know that  $U$  is contractible, and  $U^{(r)} = U \times (G/G_r)$ . The sheaf  $(\mathbb{Z}/p^s)^{Z_r}$  on  $Y_r$  consists of locally constant  $\mathbb{Z}/p^s$ -valued functions, which vanish on  $Z_r$ . It follows that  $\check{H}^\bullet(U^{(r)}, (\mathbb{Z}/p^s)^{Z_r})$  is a direct sum of finitely many copies of  $\check{H}^\bullet(U, (\mathbb{Z}/p^s)^{Z_0})$ . We must therefore show that  $\check{H}^n(U, (\mathbb{Z}/p^s)^{Z_0}) = 0$  for all  $n > 0$ .

If  $U \cap Z_0$  is empty, then we have  $\check{H}^n(U, (\mathbb{Z}/p^s)^{Z_0}) = \check{H}^n(U, \mathbb{Z}/p^s)$ . By the comparison theorem, this is the same as singular cohomology, and therefore only depends on  $U$  up to homotopy. Since  $U$  is contractible, it follows that  $\check{H}^n(U, \mathbb{Z}/p^s) = 0$  for  $n > 0$ .

Suppose instead that  $U \cap Z_0$  is non-empty. In this case, we know that  $U \cap Z_0$  is a deformation retract of  $U$ . It follows that the restriction map  $H_{\text{sing}}^\bullet(U, \mathbb{Z}/p^s) \rightarrow H_{\text{sing}}^\bullet(U \cap Z_0, \mathbb{Z}/p^s)$  is an isomorphism. By the comparison theorem, it follows that the map  $\check{H}^\bullet(U, \mathbb{Z}/p^s) \rightarrow \check{H}^\bullet(U \cap Z_0, \mathbb{Z}/p^s)$  is an isomorphism. The long exact sequence shows that  $\check{H}^\bullet(U, U \cap Z_0, \mathbb{Z}/p^s) = 0$ .

In particular, using Leray's Theorem, we have

$$\check{H}^\bullet(Y_r, Z_r, \mathbb{Z}/p^s) = \check{H}^\bullet(\mathfrak{U}^{(r)}, (\mathbb{Z}/p^s)^{Z_r}).$$

**3.3.  $\tilde{\mathfrak{U}}$  is  $\mathcal{C}$ -acyclic.** Let  $U$  be an intersection of finitely many sets in  $\mathfrak{U}$ , and let  $\tilde{U}$  be the preimage of  $U$  in  $Y$ . We know that  $U$  is contractible, and  $\tilde{U} = U \times G$ . We must show that  $\check{H}^n(\tilde{U}, \mathcal{C}) = 0$  for  $n > 0$ .

Let  $\tilde{\mathfrak{V}}$  be an open cover of  $\tilde{U}$ , and choose an element  $\sigma \in \check{H}^n(\tilde{\mathfrak{V}}, \mathcal{C})$  with  $n > 0$ . We shall find a refinement  $\tilde{\mathfrak{W}}$  of  $\tilde{\mathfrak{V}}$ , such that the image of  $\sigma$  in  $\check{H}^\bullet(\tilde{\mathfrak{W}}, \mathcal{C})$  is zero. By passing to a refinement of  $\tilde{\mathfrak{V}}$  if necessary, we may assume that  $\tilde{\mathfrak{V}}$  is finite, and that each element of  $\tilde{\mathfrak{V}}$  is of the form  $V_i \times H_i$  for some open subset  $V_i \subset U$  and some open coset  $H_i \subset G$ . By refining still further, we may assume that the cosets  $H_i$  are all cosets of the same open subgroup  $G_r \subset G$ . This means that  $\tilde{\mathfrak{V}}$  is the pullback of an open cover  $\mathfrak{V}^{(r)}$  of  $U^{(r)}$ . For an open subset  $V^{(r)} \subset U^{(r)}$ , we have

$$\mathcal{C}(\tilde{V}^{(r)}) = S(V^{(r)}),$$

where  $S$  is the locally constant sheaf on  $U^{(r)}$  with values in  $\mathcal{C}(G_r, \mathbb{Z}_p)$  and  $\tilde{V}^{(r)}$  is the preimage of  $V^{(r)}$  in  $\tilde{U}$ . It follows that

$$\check{H}^\bullet(\tilde{\mathfrak{V}}, \mathcal{C}_{\mathbb{Z}_p}) = \check{H}^\bullet(\mathfrak{V}^{(r)}, S).$$

Since  $U^{(r)}$  is homotopic to a finite set and  $S$  is a constant sheaf, it follows that  $\check{H}^{>0}(U^{(r)}, S)$  is zero. This implies there is a refinement  $\mathfrak{W}^{(r)}$  of  $\mathfrak{V}^{(r)}$ , such that the image of  $\sigma$  in  $\check{H}^\bullet(\mathfrak{W}^{(r)}, S)$  is zero. Pulling  $\mathfrak{W}^{(r)}$  back to  $\tilde{U}$ , we have a refinement  $\tilde{\mathfrak{W}}$  of  $\tilde{\mathfrak{V}}$ , such that the image of  $\sigma$  in  $\check{H}^\bullet(\tilde{\mathfrak{W}}, \mathcal{C}_{\mathbb{Z}_p})$  is zero.

**3.4.  $\tilde{\mathfrak{U}}$  is  $\mathcal{C}^Z$ -acyclic.** Let  $U$  be an intersection of finitely many sets in  $\mathfrak{U}$ , and let  $\tilde{U}$  be the preimage of  $U$  in  $Y$ . We know that  $U$  is contractible, and  $\tilde{U} = U \times G$ . We must show that  $\check{H}^n(\tilde{U}, \tilde{U} \cap Z, \mathcal{C}) = 0$  for  $n > 0$ . If  $U$  does not intersect  $Z_0$ , then this follows from the previous part of the proof. We therefore assume that  $U$  intersects  $Z_0$ . In this case, we know that  $U \cap Z_0$  is a deformation retract of  $U$ . In particular,  $U \cap Z_0$  is contractible, and  $\tilde{U} \cap Z = (U \cap Z_0) \times G$ . The previous part of the proof shows that  $\check{H}^{>0}(\tilde{U}, \mathcal{C}) = 0$  and  $\check{H}^{>0}(\tilde{U} \cap Z, \mathcal{C}) = 0$ . Furthermore, one sees immediately that the restriction map  $\check{H}^0(\tilde{U}, \mathcal{C}) \rightarrow \check{H}^0(\tilde{U} \cap Z, \mathcal{C})$  is an isomorphism. Hence by the long exact sequence, we have  $\check{H}^\bullet(\tilde{U}, \tilde{U} \cap Z, \mathcal{C}) = 0$ .

Thus by Leray's Theorem, we have:

$$\check{H}^\bullet(Y, Z, \mathcal{C}) = \check{H}^\bullet(\tilde{\mathfrak{U}}, \mathcal{C}^Z).$$

3.5. Fix for a moment a cohomological degree  $n$ , and let  $U_1, \dots, U_N$  be the non-empty intersections of  $n+1$ -tuples of sets in  $\mathfrak{U}$ , for which  $U_i \cap Z_0 = \emptyset$ . For each  $U_i$ , we let  $U_i^{(r)}$  be the preimage of  $U_i$  in  $Y_r$  and  $\tilde{U}_i$  be the preimage of  $U_i$  in  $Y$ .

Recall that  $\check{H}^\bullet(\mathfrak{U}^{(r)}, (\mathbb{Z}/p^r)^Z)$  is the cohomology of the chain complex

$$\check{C}^n(\mathfrak{U}^{(r)}, (\mathbb{Z}/p^r)^Z) = \prod_{i=1}^N (\mathbb{Z}/p^r)^{Z_r}(U_i^{(r)}),$$

Each  $U_i$  is contractible and disjoint from  $Z_0$ . Furthermore  $U_i^{(r)} = U_i \times (G/G_r)$ , so we have an isomorphism of  $G$ -modules:  $(\mathbb{Z}/p^r)^{Z_r}(U_i^{(r)}) = (\mathbb{Z}/p^r)(G/G_r)$ . This gives

$$\check{C}^n(\mathfrak{U}^{(r)}, (\mathbb{Z}/p^r)^Z) = (\mathbb{Z}/p^r)(G/G_r)^N,$$

Similarly, we have

$$\check{C}^n(\tilde{\mathfrak{U}}, (\mathbb{Z}/p^s)^Z) = \left( (\mathbb{Z}/p^s)(G) \right)^N.$$

Comparing the two formulae, it is clear that

$$\check{C}^\bullet(\tilde{\mathfrak{U}}, (\mathbb{Z}/p^s)^Z) = \varinjlim_r \check{C}^\bullet(\mathfrak{U}^{(r)}, (\mathbb{Z}/p^s)^{Z_r}).$$

Since the functor  $\varinjlim_r$  is exact, we have

$$\check{H}^\bullet(\tilde{\mathfrak{U}}, (\mathbb{Z}/p^s)^Z) = \varinjlim_r \check{H}^\bullet(\mathfrak{U}^{(r)}, (\mathbb{Z}/p^s)^{Z_r}).$$

3.6. Note also that  $\mathcal{C}^Z(\tilde{U}_i) = \mathcal{C}(G)$ , and so we have

$$\check{C}^n(\tilde{\mathfrak{U}}, \mathcal{C}^Z) = \mathcal{C}(G)^N.$$

It follows that  $\check{C}^n(\tilde{\mathfrak{U}}, \mathcal{C}^Z)$  is an admissible  $\mathbb{Z}_p[G]$ -module in the sense of [2]. Furthermore we have:

$$\check{C}^\bullet(\tilde{\mathfrak{U}}, \mathcal{C}^Z) = \varprojlim_s \check{C}^\bullet(\tilde{\mathfrak{U}}, (\mathbb{Z}/p^s)^Z), \quad \check{C}^\bullet(\tilde{\mathfrak{U}}, (\mathbb{Z}/p^s)^Z) = \check{C}^\bullet(\tilde{\mathfrak{U}}, \mathcal{C}^Z)/p^s.$$

Hence by Proposition 1.2.12 of [2], we have:

$$\check{H}^\bullet(\tilde{\mathfrak{U}}, \mathcal{C}^Z) = \varprojlim_s \check{H}^\bullet(\tilde{\mathfrak{U}}, (\mathbb{Z}/p^s)^Z).$$

By the previous part of the proof, we have:

$$\check{H}^\bullet(\tilde{\mathfrak{U}}, \mathcal{C}^Z) = \varprojlim_s \varinjlim_r \check{H}^\bullet(\mathfrak{U}^{(r)}, (\mathbb{Z}/p^s)^{Z_r}).$$

Since our covers are acyclic, this translates to

$$\check{H}^\bullet(Y, Z, \mathcal{C}) = \varprojlim_s \varinjlim_r \check{H}^\bullet(Y_r, Z_r, \mathbb{Z}/p^s).$$

On the other hand, by Theorem 5, we have

$$\check{H}^\bullet(Y_r, Z_r, \mathbb{Z}/p^s) = H_c^\bullet(Y_r \setminus Z_r, \mathbb{Z}/p^s).$$

The result follows.  $\square$

**Corollary 1.** *With the above notation,*

$$\tilde{H}^n(Y, \mathbb{Z}_p) = \check{H}^n(Y, \mathbb{C}_{\mathbb{Z}_p}), \quad \tilde{H}^n(Y, \mathbb{Q}_p) = \check{H}^n(Y, \mathbb{C}_{\mathbb{Q}_p}).$$

*Proof.* We apply the theorem in the case that  $Z$  is empty. Since  $Y^0 = Y$ , which is compact, it follows that usual cohomology is the same as compactly supported cohomology on each  $Y_r$ .  $\square$

**Corollary 2.** *In the notation of the introduction, there are long exact sequences:*

$$\begin{aligned} \tilde{H}_c^n(K^{\mathfrak{p}}, \mathbb{Z}_p) &\rightarrow \tilde{H}^n(K^{\mathfrak{p}}, \mathbb{Z}_p) \rightarrow \tilde{H}_{\partial}^n(K^{\mathfrak{p}}, \mathbb{Z}_p) \rightarrow \tilde{H}_c^{n+1}(K^{\mathfrak{p}}, \mathbb{Z}_p), \\ \tilde{H}_c^n(K^{\mathfrak{p}}, \mathbb{Q}_p) &\rightarrow \tilde{H}^n(K^{\mathfrak{p}}, \mathbb{Q}_p) \rightarrow \tilde{H}_{\partial}^n(K^{\mathfrak{p}}, \mathbb{Q}_p) \rightarrow \tilde{H}_c^{n+1}(K^{\mathfrak{p}}, \mathbb{Q}_p). \end{aligned}$$

*Proof.* For convenience, we shall write  $Y$  instead of  $Y(K^{\mathfrak{p}})$ . We have shown above that

$$\begin{aligned} \tilde{H}^{\bullet}(K^{\mathfrak{p}}, \mathbb{Q}_p) &= \check{H}^{\bullet}(Y, \mathbb{C}_{\mathbb{Q}_p}), \\ \tilde{H}_{\partial}^{\bullet}(K^{\mathfrak{p}}, \mathbb{Q}_p) &= \check{H}^{\bullet}(\partial Y^{\text{B.S.}}, \mathbb{C}_{\mathbb{Q}_p}), \\ \tilde{H}_c^{\bullet}(K^{\mathfrak{p}}, \mathbb{Q}_p) &= \check{H}^{\bullet}(Y^{\text{B.S.}}, \partial Y^{\text{B.S.}}, \mathbb{C}_{\mathbb{Q}_p}). \end{aligned}$$

There is a long exact sequence in Čech cohomology:

$$\check{H}^n(Y^{\text{B.S.}}, \mathbb{C}_{\mathbb{Q}_p}) \rightarrow \check{H}^n(\partial Y, \mathbb{C}_{\mathbb{Q}_p}) \rightarrow \check{H}^{n+1}(Y^{\text{B.S.}}, \partial Y^{\text{B.S.}}, \mathbb{C}_{\mathbb{Q}_p}) \rightarrow \check{H}^{n+1}(Y^{\text{B.S.}}, \mathbb{C}_{\mathbb{Q}_p}).$$

The same holds for  $\mathbb{Z}_p$ .  $\square$

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